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Comment on Quarks and Gluons at small x and the SDIS Factorization Scheme *

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Abstract

I present some comments on the partonic interpretation of the HERA data on the proton structure function. The effects of the resummation of the leading and next-to-leading $\ln x$-contributions are discussed. A new factorization scheme, in which these resummation effects are absorbed into a steep redefinition of the gluon density, is introduced and its (possible) interpretation and phenomenological relevance are suggested.

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1. Introduction

The electron-proton collider HERA has opened up a new kinematic regime in the study of the deep structure of the proton and, in general, of hadronic interactions. This regime is characterized by large values of momentum transfer Q ($Q \gtrsim 1$ GeV) and increasing centre-of-mass energy $\sqrt{S}$ or, equivalently, by small values of the Bjorken variable $x = Q^2/S$. The first experimental results from HERA in 1992 have shown a striking rise of the proton structure function $F_2(x, Q^2)$ for values $x < 10^{-2}$. The observation of this strong increase of F_2 has been then confirmed by the following and more precise data [1].

The widespread interest in the HERA results on F_2 is not simply due to the fact that they represent the first experimental observation of a cross section increasing faster than logarithmically with the energy (see, for instance, Ref. [2]). More importantly, in fact, they have attracted much theoretical attention because the rise of F_2 at low x can be a first signal of non-conventional QCD dynamics [3].

At present, there are essentially two quite general theoretical approaches which aim to explaining this increase of the proton structure function. The first approach [4-9] is based on conventional perturbative QCD. Here the parton densities of the proton at a fixed input scale Q_0^2 are evolved in Q^2 according to the Altarelli-Parisi equation [10] evaluated in *fixed-order* perturbation theory. I refer to this approach as conventional because it has been successfully applied and tested in the region of moderate and large values of x [5-7,11]. The second approach [12-15], based either on the original BFKL equation [16] or on the high-energy (or $k_\perp$-) factorization [17-22] is less conventional. It is motivated by the observation that, at asymptotically small values of x, the fixed-order pertubative expansion in the strong coupling α_S must become inadequate to describe the QCD dynamics. Indeed, multiple gluon radiation in the final state produces logarithmic corrections of the type $(\alpha_S \ln x)^n$: as soon as x is sufficiently small (i.e. $\alpha_S \ln 1/x \sim 1$), these terms have to be resummed to all orders in α_S in order to get reliable theoretical predictions.

The investigations carried out during the last two years [5-9,12] have shown that both approaches can produce phenomenological results in agreement with the rise of F_2 as observed at HERA. In particular, the conventional perturbative-QCD approach is very successful in describing the main features of HERA data and, hence, the signal of non-conventional QCD dynamics (at least from F_2, in the kinematic region explored at HERA so far) is *hidden* or *mimicked* by a strong background of conventional QCD evolution.

The present situation thus demands data which are more accurate and cover a larger phase space region both in x and Q^2. At the same time, however, theoretical progress is urgently required. The main theoretical issue we have to face is indeed the following. On one side, the conventional perturbative-QCD approach is very much well founded and hence, in a sense, privileged. On the other side, the approach based on small-x resummation has been fully set up only in leading order (i.e. resummation of the terms $(\alpha_S \ln x)^n$) and, hence, it suffers form large theoretical uncertainties as in any leading-order analysis. In this respect, we thus need a more refined theory [22,23] and, in particular, the calculation of *all* the next-to-leading corrections $\alpha_S(\alpha_S \ln x)^n$.

In this letter, I am not going to present any new theoretical or phenomenological result. Starting from the actual knowledge of part of the next-to-leading corrections at small-x

[21,22], I shall limit myself to make a few comments which may contribute to the present discussion on the interpretation of the HERA data on F_2. These comments refer, in general, to the parton language. Although, possibly, we should eventually abandon this language to improve our understanding of the non-perturbative QCD region (in particular, the behaviour of $F_2(x,Q^2)$ in the transition from low to high values of Q^2), it is certainly true that the partonic picture is nowadays privileged as for the interpretation of the hadronic interactions in the hard-scattering regime. Therefore, in this context, I shall try to address two main points.

Firstly, I would like to recall that, although the concept of parton is qualitatively very simple (i.e. a parton is a point-like constituent of the proton), the parton (quark and gluon) densities are not physical observables. In fact, they have a physical meaning only within a given (and well-defined) theoretical framework.

Secondly, once the theoretical framework has been specified, I shall try to arise the question whether we can understand (explain) the small-x behaviour of the quark and gluon densities.

The outline of the paper is as follows. In Sect. 2, I first recall the general framework of the conventional QCD approach to the scaling violations of $F_2(x,Q^2)$. Then, I summarize the ensuing results for the small-x behaviour of the proton parton densities. Section 3 is devoted to review the present theoretical status of small-x resummation. The resummed results presented in this Section are then discussed in Sec. 4 in the context of the scaling violations of $F_2(x,Q^2)$ and of the determination of the parton densities. In Sec. 5, I introduce a new factorization scheme in which the resummation effects considered above are completely embodied in the redefinition of the gluon density. Some final comments are left to Sec. 6.

2. Proton structure function and parton densities

A theoretical framework (the only one, as far as I know!) in which quark and gluon densities are unambiguously † defined is that provided by the (QCD) factorization theorem of mass singularities [24,25]. Here one starts from the leading-twist expansion of a certain physical observable and considers its perturbative QCD evolution in terms of generalized Altarelli-Parisi equation.

To be definite, let me consider the so called DIS factorization scheme [26]. In this scheme, the master equations for the proton structure function at small x are as follows

$$\begin{aligned} F_2(x,Q^2) &= <e_f^2> \tilde{f}_S(x,Q^2) + \ \ldots \ + O(1/Q^2) \ , \qquad (1)\\ \frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} &= <e_f^2> \int_x^1 dz \Big[P_{SS}(\alpha_S(Q^2),z)\, \tilde{f}_S\left(x/z,Q^2\right) \\ &+ \ P_{Sg}(\alpha_S(Q^2),z)\, \tilde{f}_g\left(x/z,Q^2\right)\Big] + \ \ldots \ + O(1/Q^2) \ , \qquad (2) \end{aligned}$$

where e_f is the electric charge of each quark with flavour f, $< e_f^2 >= (\sum_{f=1}^{N_f} e_f^2)/N_f$ and N_f is the number of active flavours. In Eqs. (1),(2) I am using the same notation as in

†By unambiguously I mean defined to *any* order in α_S and with full control of the *factorization scheme dependence*.

Ref. [17]. Thus, the singlet density $\tilde{f}_S$ and the gluon density $\tilde{f}_g$ are related to the usual quark (antiquark) and gluon densities f_{q_f} ($f_{\bar{q}_f}$) and f_g by the following relations

$$\tilde{f}_S(x,Q^2) = x \sum_f \left[f_{q_f}(x,Q^2) + f_{\bar{q}_f}(x,Q^2) \right] \ , \quad \tilde{f}_g(x,Q^2) = x f_g(x,Q^2) \ , \tag{3}$$

and the quark splitting function P_{SS} and P_{Sg} are given in terms of the customary Altarelli-Parisi splitting functions P_{ab} as follows

$$P_{Sg}(\alpha_S,x) = 2N_f P_{q_i g}(\alpha_S,x) \ , \quad P_{SS}(\alpha_S,x) = \sum_j \left[P_{q_i q_j}(\alpha_S,x) + P_{q_i \bar{q}_j}(\alpha_S,x) \right] \ . \tag{4}$$

The dots and the terms $O(1/Q^2)$ on the right-hand side of Eqs. (1),(2) denote respectively the flavour non-singlet component (which is numerically negligible at small-x) and higher-twist contributions.

Note that Eq. (1) actually represents the definition of the singlet-quark density $\tilde{f}_S$. The true dynamical information is instead contained in the scaling violations as described by Eq. (2) and by the analogous evolution equation for the gluon density, namely:

$$\begin{aligned} \frac{d\tilde{f}_g(x,Q^2)}{d\ln Q^2} &= \int_x^1 dz \left[P_{gq}(\alpha_S(Q^2),z) \ \tilde{f}_S\left(x/z,Q^2\right) \right. \\ &+ \left. P_{gg}(\alpha_S(Q^2),z) \ \tilde{f}_g\left(x/z,Q^2\right) \right] \ . \end{aligned} \tag{5}$$

Note also that the Altarelli-Parisi splitting functions entering into Eqs. (2),(5) are computable in QCD perturbation theory as a power series expansion in α_S:

$$P_{ab}(\alpha_S,x) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{2\pi} \right)^n \ P_{ab}^{(n-1)}(x) \ , \tag{6}$$

and the coefficients $P_{ab}^{(n-1)}(x)$ in this series can be calculated (at least, in principle) to any order n in α_S.

In the conventional QCD analyses carried out at present, *only* the first two non-trivial terms $P_{ab}^{(0)}(x)$ and $P_{ab}^{(1)}(x)$ are taken into account. Then, by using the experimental information on F_2 and $dF_2/d\ln Q^2$, one can (self-)consistently determine the quark and gluon densities as functions of x at a certain input scale Q_0^2. I do not want to discuss the (although relevant) differences among the detailed analyses carried out by the various authors. The main points that I would like to recall are the typical results [‡] of this conventional QCD approach. Assuming the following small-x behaviour of the parton densities

$$\tilde{f}_S(x,Q_0^2) \simeq x^{-\lambda_S} \ , \quad \tilde{f}_g(x,Q_0^2) \simeq x^{-\lambda_g} \tag{7}$$

and *imposing* the constraint $\lambda_S = \lambda_g$, one finds [5-7,27] $\lambda_S = \lambda_g = 0.2 \div 0.3$ at the input scale $Q_0^2 \sim 4$ GeV^2. More recently, it has been pointed out that a better (self-)consistent description of the HERA data can be achieved by *relaxing* the constraint $\lambda_S = \lambda_g$: in this case, at the same input scale $Q_0^2 \sim 4$ GeV^2, one finds [8] the following best-fit values

$$\lambda_S = 0.07 \ , \quad \lambda_g = 0.3 \div 0.35 \ . \tag{8}$$

[‡] To be precise, the values of λ_S and λ_g reported below refer the $\overline{\mathrm{MS}}$ factorization scheme. However, to this order in perturbation theory, the $\overline{\mathrm{MS}}$ and DIS schemes give very similar quantitative results.

Two comments are in order.

i) Independently of the actual values of λ_S and λ_g, the power behaviour in Eq. (7) calls forth an interpretation in terms of the BFKL approach, or, in general, in terms of the non-conventional QCD approach based on small-x resummation. *ii*) Taking seriously the results of the MRS(G) analysis [8] in Eq. (8), one can argue that $F_2(x,Q^2)$ is *not* very steep at Q^2-values of the order of few GeV^2, but it is driven by *strong* scaling violations. As a matter of fact, F_2 gives information on the sea quark density $\tilde{f}_S$ (see Eq. (1)) which, according to the value of λ_S in Eq. (8), is pretty flat for relatively small values of Q^2. Then, having fixed $\tilde{f}_S$, $\partial F_2/\partial \ln Q^2$ gives information on the product (convolution) $P_{Sg} \otimes \tilde{f}_g$ (see Eq. (2)). In the conventional QCD analysis, only the first two orders in the perturbative series (4) for $P_{Sg}(\alpha_S,x)$ are considered and, since they are not very singular at small x, the large value of λ_g in Eq. (8) is necessary to account for a steep behaviour of $\partial F_2/\partial \ln Q^2$.

Note however that, strictly speaking, the measurement of $\partial F_2/\partial \ln Q^2$ does not give access directly to the determination of the gluon density $\tilde{f}_g$, but rather to that of the product $P_{Sg} \otimes \tilde{f}_g$. I shall comment more on this point in the following Sections. For the moment, let me come back to discuss the Lipatov-like behaviour in Eq. (7).

The BFKL equation [16] predicts a universal power-like increase of the hadronic cross sections with the energy. In the case of the proton structure function, this implies the behaviour $x^{-\lambda_L}$, where $1+\lambda_L = 1 + 4\bar{\alpha}_S \ln 2 \simeq 1 + 2.65\alpha_S$ ($\bar{\alpha}_S = C_A\alpha_S/\pi$) is known as the intercept of the perturbative QCD pomeron. Obviously, it is quite difficult to extract definite quantitative predictions from this theoretical analysis: within the present leading-order formalism there is no control on the scale of α_S (and, hence, on the precise value of α_S) and on the size of the $O(\alpha_S^2)$-corrections in the expression for λ_L. However, a point which I would like to address is that in the BFKL analysis the increase of the cross section is simply due to muliple gluon radiation. Therefore, according to the common wisdom, the gluon channel is dominant and the quark density is simply driven by the gluon density. It follows that one may expect a power behaviour as in Eq. (7) with $\lambda_S = \lambda_g$. In this respect, it is thus difficult to explain why the HERA data may prefer [8] a value $\lambda_S < \lambda_g$, as given in Eq. (8).

In the following section I shall try to explain that this common wisdom can actually be too naïve because it overlooks the meaning of parton densities, as given by the factorization theorem of mass singularities.

3. High-energy factorization and small-x resummation

As discussed above, only gluons [§] enter in the leading-order BFKL approach. How do quarks can be included in a framework aimed to go beyond the conventional perturbative QCD picture?

A formalism which is able to combine *consistently* the BFKL equation (and, in general, small-x resummation) with the factorization theorem of mass singularities has been set up in the last few years. This formalism, known as $k_\perp$-factorization or high-energy

[§]Strictly speaking, within the original BFKL framework, even the concept of gluon density is meaningless.

factorization, was first discussed to leading-order accuracy in Refs. [17-19] and then was extended to higher-orders in Refs. [20,22]. In the high-energy factorization approach, the resummation of the $\ln x$-corrections embodied by the BFKL equation is translated into the parton language by performing a leading-twist expansion which is consistent with QCD collinear factorization [25]. As a result, one is dealing with the usual QCD evolution equations (namely, Eqs. (1), (2) and (5) in the case of the proton structure function F_2) but the Altarelli-Parisi splitting functions $P_{ab}(\alpha_S, x)$ (and, in general, the process dependent coefficient functions) are no longer evaluated in fixed-order perturbation theory. They are indeed supplemented with the all-order resummation of the leading $(\frac{1}{x}\alpha_S^n \ln^{n-1} x)$, next-to-leading $(\frac{1}{x}\alpha_S^n \ln^{n-2} x)$ and, possibly, subdominant $(\frac{1}{x}\alpha_S^n \ln^m x,\ m < n-2)$ contributions at small x.

According to the $k_\perp$-factorization picture, the proton structure function F_2 is obtained by coupling the off-shell photon to the BFKL gluon distribution via a quark loop [¶] (Fig.1). The BFKL distribution resums perturbative contributions of the type $\frac{1}{x}\alpha_S^n \ln^{n-1} x$. These are associated to the emission of gluons, with any value of transverse momentum $k_\perp$, over the large rapidity gap $\Delta y = \ln 1/x$. In other words, no $k_\perp$-ordering is imposed on the gluon evolution and, consistently, no $k_\perp$-ordering is enforced by coupling the BFKL distribution to the quark loop. Indeed, the quark box contribution has to be evaluated by keeping off-shell the incoming gluon k ($k^2 \simeq -k_\perp^2 \neq 0$).

Because of the absence of $k_\perp$-ordering the partonic interpretation of the $k_\perp$-factorization picture has to be considered with care. In particular, one cannot simply argue that the quark box is not singular at small x (the exchange of a spin 1/2 particle in the t-channel leads to a vanishing amplitude in the high-energy limit) and, hence, the sea quark distribution is driven by the gluon distribution. This common wisdom is too naïve and, possible, misleading.

Since there is no $k_\perp$-ordering, there are two relevant integration regions in Fig.1: a) $Q^2 \sim k_\perp'^2 \gg k_\perp^2$ and b) $Q^2 \gg k_\perp'^2 \sim k_\perp^2$. In the region a) only the gluon $k_\perp$ (and not the quark) can approach the mass-shell and thus the sea quark provides an effective coupling between the off-shell photon and the gluon density. In the region b), instead, also the quark $k'_\perp$ can be close to the mass-shell and in this case the off-shell photon is probing the Q^2-evolution of the sea quark density.

According to the QCD factorization theorem of mass singularities the actual separation (which is mandatory for any consistent partonic interpretation) between the two phase-space regions a) and b) is *factorization scheme dependent*. Indeed, the region b) produces collinear singularities to any order in α_S when $k_\perp'^2 \simeq k_\perp^2 \to 0$. Therefore one has to specify the factorization scheme (procedure) in order to define what is the gluon density $\tilde{f}_g$ and what is the quark density $\hat{f}_S$ beyond the leading order.

The detailed analysis of this issue and the ensuing explicit calculations in different factorization schemes (namely, the $\overline{\rm MS}$ and DIS schemes) have been performed in Ref. [20-22]. Let me recall some of the main outcomes of these studies.

[¶]In this paper I limit myself to a qualitative description. The full formalism is discussed in detail in Refs. [17,22].

To this purpouse, it is convenient to introduce the anomalous dimensions $\gamma_{ab,N}(\alpha_S)$, that is, the N-moments of the Altarelli-Parisi splitting functions:

$$\gamma_{ab,N}(\alpha_S) \equiv \int_0^1 dx\, x^N P_{ab}(\alpha_S, x) \ . \tag{9}$$

Note that logarithmic contributions of the type $\frac{1}{x}\ln^{n-1} x$ in x-space correspond to multiple poles $(1/N)^n$ in N-space.

The scheme dependence of the gluon density $\hat{f}_g$ was discussed in detail in Ref. [20] and then in Ref. [22]. In particular, it was shown that the resummation of the leading terms $\frac{1}{x}\alpha_S^n \ln^{n-1} x$ ($(\alpha_S/N)^n$ in N-space) in the gluon splitting function $P_{gg}(\alpha_S, x)$ leads to the celebrated BFKL anomalous dimension $\gamma_N(\alpha_S)$ [16], that is,

$$\gamma_{gg,N}(\alpha_S) = \gamma_N(\alpha_S) + \mathcal{O}\left(\alpha_S(\alpha_S/N)^n\right) \ . \tag{10}$$

Here, $\gamma_N(\alpha_S)$ is obtained by solving the implicit equation ($\bar{\alpha}_S = C_A\alpha_S/\pi$)

$$1 = \frac{\bar{\alpha}_S}{N}\, \chi\left(\gamma_N(\alpha_S)\right) \ , \tag{11}$$

where the characteristic fuctions $\chi(\gamma)$ is expressed in terms of the Euler ψ-function as follows

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \ . \tag{12}$$

Note that the results in Eqs. (10)-(12) were first derived in Refs. [20,22] by consistently carrying out the procedure of factorization of the collinear singularities in dimensional regularization. In this way we could address properly the issue of the scheme dependence of $\tilde{f}_g$ and P_{gg}. In particular, we were able to show that Eq. (10) is actually valid in the $\overline{\rm MS}$ and DIS schemes and, in general, in any factorization scheme which does not introduce pathologically ‖ singular terms of the type α_S^n/N^{n+p} ($p \geq 1$) in the perturbative calculation at high energy **.

In Ref. [22], it was also shown that the non-diagonal gluon anomalous dimension $\gamma_{gq,N}(\alpha_S)$, when evaluated in resummed perturbation theory, is related to $\gamma_{gg,N}(\alpha_S)$ by the following colour charge relation

$$\gamma_{gg,N}(\alpha_S) = \frac{C_F}{C_A}\, \gamma_{gg,N}(\alpha_S) + \mathcal{O}\Big(\alpha_S \left(\frac{\alpha_S}{N}\right)^n\Big) = \frac{C_F}{C_A}\, \gamma_N(\alpha_S) + \mathcal{O}\Big(\alpha_S \left(\frac{\alpha_S}{N}\right)^n\Big) \ . \tag{13}$$

Note, again, that Eq. (13) is not scheme independent. It is valid in the $\overline{\rm MS}$ and DIS schemes but *can be violated* in many other factorization schemes (see the discusssion in Sect. 5) in which Eq. (10) is still true!

The next-to-leading resummed contributions $\alpha_S\,(\alpha_S/N)^n$ in Eqs. (10),(13) are not yet known beyond two-loop order ($n = 1$). However, the analogous contributions to the quark splitting functions $P_{Sg}(\alpha_S, x), P_{SS}(\alpha_S, x)$ (or, anomalous dimensions $\gamma_{Sg,N}(\alpha_S), \gamma_{SS,N}(\alpha_S)$)

‖In those schemes where such terms are present, they cancel by combining coefficient function and anomalous dimension contributions.

**This feature of the gluon anomalous dimensions was first pointed out by T. Jaroszewicz [28].

were computed in Refs. [21,22]. These contributions are the most singular in the quark sector (due to the gluon dominance at high-energy, terms of the type $(\alpha_S/N)^n$ are absent both in $\gamma_{Sg,N}$ and $\gamma_{SS,N}$) but, despite this fact, they (and the corresponding sea quark density $\tilde{f}_S$!) are *not* factorization scheme independent. Since in Sect. 2 I have simplified the discussion on F_2 by limiting myself to consider the DIS scheme, I shall recall the results in this scheme [††].

The resummed expression for the quark anomalous dimension $\gamma_{Sg,N}(\alpha_S)$ is the following

$$\gamma_{Sg,N}(\alpha_S) = h_2(\alpha_S, \gamma_N(\alpha_S)) \; R(\gamma_N(\alpha_S)) + \mathcal{O}\left(\alpha_S^2 \, (\alpha_S/N)^n\right) \; , \tag{14}$$

where the functions $h_2(\alpha_S, \gamma)$ and $R(\gamma)$ are given by [21]

$$h_2(\alpha_S, \gamma) = \frac{\alpha_S}{2\pi} T_R N_f \, \frac{2(2+3\gamma-3\gamma^2)}{3-2\gamma} \, \frac{\Gamma^3(1-\gamma)\Gamma^3(1+\gamma)}{\Gamma(2-2\gamma)\Gamma(2+2\gamma)} \; , \tag{15}$$

$$R(\gamma) = \left\{ \frac{\Gamma(1-\gamma)\,\chi(\gamma)}{\Gamma(1+\gamma)[-\gamma\,\chi'(\gamma)]} \right\}^{\frac{1}{2}} \exp\left\{ \gamma\psi(1) + \int_0^\gamma dx \frac{\psi'(1)-\psi'(1-x)}{\chi(x)} \right\} \; , \tag{16}$$

and χ and χ' are the characteristic function in Eq. (12) and its first derivative, respectively.

Equation (14) resums all the perturbative corrections of the type $\alpha_S \, (\alpha_S/N)^n$. This resummation is achieved through the γ-dependence of h_2 and R in Eqs. (15),(16) and the (α_S/N)-dependence of the BFKL anomalous dimension $\gamma_N(\alpha_S)$ in Eq. (14).

In the DIS scheme, the anomalous dimension $\gamma_{SS,N}(\alpha_S)$ is related to $\gamma_{Sg,N}(\alpha_S)$ by a colour charge relation analogous to Eq. (13):

$$\gamma_{SS,N}(\alpha_S) = \frac{C_F}{C_A} \left[\gamma_{Sg,N}(\alpha_S) - \frac{\alpha_S}{2\pi} \frac{4}{3} T_R N_f \right] + \mathcal{O}\left(\alpha_S^2 \left(\frac{\alpha_S}{N} \right)^n \right) \; . \tag{17}$$

4. Scaling violations at small x

The amount of pertubative scaling violation in the proton structure function F_2 is controlled by the Altarelli-Parisi splitting funtions via the Eqs. (1),(2),(5). In Sect. 2, I have recalled the implications of the scaling violations observed at HERA if the splitting functions are evaluated in fixed-order (more precisely, in two-loop order) perturbation theory. In this Section, I discuss the impact of small-x resummation.

Let me start by considering the evolution of the gluon density $\tilde{f}_g$ in Eqs. (5). The resummation of the leading logarithmic contributions $\frac{1}{x}\alpha_S^n \ln^{n-1} x$ in the splitting functions $P_{gg}(\alpha_S, x)$, $P_{gq}(\alpha_S, x)$ leads to consider the BFKL anomalous dimension $\gamma_N(\alpha_S)$ in Eqs. (11). Its power series expansion in α_S reads as follows

$$\gamma_N(\alpha_S) = \sum_{n=1}^{\infty} C_n \left(\frac{\bar{\alpha}_S}{N} \right)^n \simeq \frac{\bar{\alpha}_S}{N} + 2.404 \left(\frac{\bar{\alpha}_S}{N} \right)^4 + 2.074 \left(\frac{\bar{\alpha}_S}{N} \right)^6 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N} \right)^7 \right) \; . \tag{18}$$

[††] The result in the $\overline{\rm MS}$ scheme can be found in Ref. [22]. In particular, in the $\overline{\rm MS}$ scheme the quark anomalous dimensions are smaller but they are compensated by a corresponding enhancement in the coefficient functions (see, Eq. (5.38) in Ref. [22].)

Note that most of the first coefficients in this expansion is vanishing. This implies that the deviations from the fixed-order expansion are expected to be small, at least for moderate value of $\ln 1/x$.

All the other coefficients in the expression (18) are of the order of two. Actually, the characteristic function $\chi(\gamma)$ in Eq. (12) has approximately a parabolic shape. As $x \to 0, N$ decreases and reaches a minimum value $N_{\rm min} = \lambda_L = 4\bar{\alpha}_S \ln 2 \simeq 2.65\, \alpha_S$ at which γ_N has a branch point singularity. Therefore the resummation of the singular terms $(\alpha_S/N)^n$ builds up a stronger singularity at $N = \lambda_L$. This singularity, known as the perturbative QCD (or BFKL) pomeron, is responsible for the following asymptotic behaviour of the gluon splitting functions ($\zeta(3) \simeq 1.202$)

$$\begin{aligned} P_{gg}(\alpha_S, x)|_{\rm asym.} &\simeq \frac{C_A}{C_F} P_{gq}(\alpha_S, x)|_{\rm asym.} \simeq \frac{1}{\sqrt{56\pi\,\zeta(3)}} \frac{\bar{\alpha}_S}{x}\, x^{-\lambda_L} \left(\bar{\alpha}_S \ln \frac{1}{x}\right)^{-\frac{3}{2}} \\ &\simeq 0.0688\, \frac{\bar{\alpha}_S}{x}\, x^{-\lambda_L} \left(\bar{\alpha}_S \ln \frac{1}{x}\right)^{-\frac{3}{2}} \quad . \end{aligned} \tag{19}$$

Note, however, that the steep behaviour in Eq. (19) is valid in the asymptotic limit $\alpha_S \ln 1/x \gg 1$. The subasymptotic corrections for $\alpha_S \ln 1/x \sim 1$ are pretty large even when $\ln \frac{1}{x} > 1$ and strongly suppress the steep behaviour in Eq. (19). This effect of the subasymptotic corrections is consistent with the slow departure of the series (18) from its one-loop truncation.

Recent numerical analyses [13,14,15] on the evolution of the gluon density in resummed perturbation theory confirm this qualitative expectation. The resummation of the leading terms $\frac{1}{x}\alpha_S^n \ln^{n-1} x$ in the gluon splitting functions has a moderate effect on the scaling violations of the proton structure function $F_2(x, Q^2)$ in the kinematic range presently investigated at HERA.

Let me now consider the quark channel. The resummed anomalous dimension in Eq. (14) has the following pertubative expansion

$$\begin{aligned} \gamma_{Sg,N}(\alpha_S) &\simeq \frac{\alpha_S}{2\pi}\, T_R N_f\, \frac{4}{3} \left\{ 1 + 2.17 \frac{\bar{\alpha}_S}{N} + 2.30 \left(\frac{\bar{\alpha}_S}{N}\right)^2 + 8.27 \left(\frac{\bar{\alpha}_S}{N}\right)^3 + \right. \\ &+ \left. 14.92 \left(\frac{\bar{\alpha}_S}{N}\right)^4 + 29.23 \left(\frac{\bar{\alpha}_S}{N}\right)^5 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^6\right) \right\} \quad . \end{aligned} \tag{20}$$

Note some main features of Eq. (20): all the perturbative coefficients are *non-vanishing*, *positive definite* and *large*.

The fact that they are non-vanishing has to be contrasted with the opposite behaviour in the BFKL anomalous dimension of Eq. (18). Therefore, in the quark sector one expects [21] a quicker departure from fixed-order perturbation theory.

The properties of the coefficients of being positive and large are also non-accidental. They have indeed a physical origin. The positivity follows from the fact that the resummed expression (14) has a probabilistic interpretation. It is derived by performing the

convolution ($k_\perp$-factorization) of the BFKL gluon distribution with a generalized (off-shell) Altarelli-Parisi (positive definite) probability [22]. Two different effects combine each other to give large perturbative coefficients. Indeed, there are large contributions coming both from the factor h_2 and from the factor R in Eq. (14). The large coefficients in h_2 are due to the logarithmically enhanced $k_\perp$-tail of the quark box diagram (see Ref. [17] for a similar discussion in the case of heavy-flavour production), whilst those in R are related to the broad $k_\perp$-spectrum of the BFKL gluon distribution [17,20].

Although the perturbative coefficients in the series (20) are much larger than those in the series (18), the resummation of the next-to-leading corrections in the quark channel does not introduce any N-plane singularity above the BFKL singularity at $N = \lambda_L = 4\bar{\alpha}_S \ln 2$. More precisely, the broadening of the $k_\perp$-spectrum of the BFKL gluon distribution (i.e., the factor R in Eq. (14)) produces a branch point singularity at $N = \lambda_L$ also for the quark anomalous dimension $\gamma_{Sq,N}(\alpha_S)$ [21]. The corresponding asymptotic expression for the quark splitting functions are:

$$
\begin{aligned}
P_{Sg}(\alpha_S,x)|_{\text{asym.}} &\simeq \frac{C_A}{C_F} P_{SS}(\alpha_S,x)|_{\text{asym.}} \simeq \frac{4\sqrt{\ln 2} K\, h_2(\alpha_S,\gamma=1/2)}{[56\zeta(3)]^{\frac{1}{4}}\,\Gamma(\frac{1}{4})}\ \frac{\bar{\alpha}_S}{x}\ x^{-\lambda_L} \left(\bar{\alpha}_S \ln\frac{1}{x}\right)^{-\frac{3}{4}} \\
&\simeq 0.0859\ \alpha_S N_f \frac{\bar{\alpha}_S}{x}\ x^{-\lambda_L} \left(\bar{\alpha}_S \ln\frac{1}{x}\right)^{-\frac{3}{4}} \quad , \qquad (21)
\end{aligned}
$$

where

$$
K = \exp\left\{ \frac{1}{2}\psi(1) + \int_0^{\frac{1}{2}} d\gamma \frac{\psi'(1) - \psi'(1-\gamma)}{\chi(\gamma)} \right\} = 0.6317. \qquad (22)
$$

The asymptotic result in Eq. (21) is formally subleading (i.e. suppressed by a power of α_S) with respect to the asymptotic behaviour in Eq. (19). However, if one considers the ratio

$$
\frac{P_{Sg}(\alpha_S,x)}{P_{gg}(\alpha_S,x)}|_{\text{asym.}} \simeq 1.249\ \alpha_S N_f \left(\bar{\alpha}_S \ln\frac{1}{x}\right)^{\frac{3}{4}} \quad , \qquad (23)
$$

one can easily notice that the asymptotic expression of $P_{Sg}(\alpha_S,x)$ can be numerically comparable to that of $P_{gg}(\alpha_S,x)$ as soon as $\alpha_S \ln 1/x \sim 1$.

Equations (21) and (23) as to be regarded mainly as a numerical exercise in the asymptotic regime $\alpha_S \ln 1/x \gg 1$. Nevertheless, two main features resulting from the resummation in the quark channel have to be emphasized. First, the resummed quark splitting functions $P_{Sg}(\alpha_S,x)$ and $P_{SS}(\alpha_S,x)$ are steeper than their fixed-order perturbative expansions. Second, this steep behaviour sets in earlier than in $P_{gg}(\alpha_S,x)$ and $P_{gq}(\alpha_S,x)$ because the perturbative coefficients in Eq. (20) are much larger than those in Eq. (18). Due to these reasons, *stronger scaling violations* at small x, coming from quark evolution, were anticipated in Ref. [21].

Let me thus come back to the comparison with the scaling violation observed at HERA, that is, to the master equations (1),(2). As discussed in Sect. 2, the large value of $\partial F_2(x,Q^2)/\partial \ln Q^2$ at small x calls for a quite steep product (convolution) $P_{Sg} \otimes \tilde{f}_g$. In the conventional (fixed-order) perturbative analysis this condition can be fulfilled only by choosing a quite steep input distribution $\tilde{f}_g$. This picture, however, can change once the

resummation of the $\ln x$-corrections in the quark channel is taken into account. The discussion of this Section shows that the resummation of the next-to-leading * terms $\frac{1}{x}\alpha_S^n \ln^{n-2} x$ leads to quark splitting functions $P_{Sg}(\alpha_S, x)$ and $P_{SS}(\alpha_S, x)$ which are much steeper than the corresponding splitting functions evaluated to the first few orders in perturbation theory. Therefore, the use of resummed perturbation theory at small x may explain the scaling violations observed at HERA without the necessity of introducing a very steep input gluon density $\tilde{f}_g$. The results of the recent numerical analysis in Ref. [13], carried out by using the resummed expression (14), support this conclusion†. Similar results have been obtained in Ref. [14].

There is also an alternative (and, possibly, more striking) way to restate the same conclusion on the relevance of small-x resummation for the HERA data on F_2. So far, I have only considered the DIS (and $\overline{\mathrm{MS}}$) factorization scheme. In the next Section, I shall introduce a new factorization scheme in which all the small-x resummed corrections in the *quark* (and not gluon!) channel are removed from P_{Sg}, P_{SS} and absorbed into the redefinition of the gluon density $\tilde{f}_g$ (and not the quark density $\tilde{f}_S$!). In the new scheme, a steep gluon density and, in particular, a gluon density steeper than the quark density arises naturally as the result of small-x resummation. Therefore, this scheme may offer a qualitative interpretation of the results of the MRS(G) analysis [8] discussed in Sect. 2.

5. The SDIS factorization scheme

In the previous Sections I have repeatedly noted that the parton densities are not physical observables. Therefore, starting from the parton densities $\tilde{f}_a$ in the DIS scheme, one can define a new set $\tilde{f}_a^{(\mathrm{SDIS})}$ of parton densities via the invertible transformation‡

$$\tilde{f}_{a,\,N}^{(\mathrm{SDIS})}(Q^2) = \sum_b U_{ab,\,N}(\alpha_S(Q^2)) \; \tilde{f}_{b,\,N}(Q^2) \quad . \tag{24}$$

The 'singular' DIS (SDIS) scheme which I am going to introduce is obtained by choosing the matrix U in such a way that

$$\tilde{f}_{q_f,\,N}^{(\mathrm{SDIS})} = \tilde{f}_{q_f,\,N} \tag{25}$$

$$\tilde{f}_{g,\,N}^{(\mathrm{SDIS})} = U_{gg,\,N}(\alpha_S) \; \tilde{f}_{g,\,N} + U_{gS,\,N}(\alpha_S) \; \tilde{f}_{S,\,N} \quad . \tag{26}$$

Equation (25) implies that the quark densities in the new scheme are the same as in the DIS scheme. The two entries U_{gg} and U_{gS}, which define the new gluon density, are

*Note that these terms are actually corrections of relative order $\alpha_S^n \ln^n x$ with respect to the splitting function P_{Sg} evaluated in two-loop order $\left(P_{Sg} \sim \alpha_S + \alpha_S^2/x\right)$. In other words, these contributions give *leading-order* corrections on the right-hand side of Eq. (2).

† The first relevant phenomenological study of F_2 by using the $k_\perp$-factorization approach [17] was performed by AKMS [12]. Their conclusions were similar to those in Refs. [13] and [14]. Nonetheless, at that time, the relation between $k_\perp$-factorization and all-order collinear factorization had not yet been fully clarified. In this respect, the method and the numerical results in Ref. [12] are still in the context of the original BFKL approach.

‡I am using the same notation as in Sec. 5.3 of Ref. [22]. Here one can find more details on factorization scheme transformations to all orders in α_S.

perturbative series in α_S which contain terms which are at most as singular as $(\alpha_S/N)^n$ for $N \to 0$. This constraint implies that the new diagonal gluon anomalous dimension $\gamma_{gg,\,N}^{(\mathrm{SDIS})}(\alpha_S)$ is still equal to the BFKL anomalous dimension to leading order in $(\alpha_S/N)^n$. However, as stated in Sect. 3, this sole constraint is *not* sufficient to guarantee the validity of the colour charge relation in Eq. (13) to the same accuracy. If one does not want to introduce other leading-order contributions in $\gamma_{gq,\,N}$, one must impose the relation

$$U_{gS,\,N}(\alpha_S) = \frac{C_F}{C_A}\ \left[\, U_{gg,\,N}(\alpha_S) - 1\right] \ . \tag{27}$$

Finally, $U_{gg,\,N}$ is given in terms of the resummed quark anomalous dimension $\gamma_{Sg,\,N}(\alpha_S)$ in Eq. (14) as follows[§]

$$U_{gg,\,N}(\alpha_S) = \frac{\gamma_{Sg,\,N}(\alpha_S)}{\gamma^{(0)}_{Sg,\,N}(\alpha_S)} + \mathcal{O}\left(\alpha_S \left(\frac{\alpha_S}{N}\right)^n\right) \ , \tag{28}$$

where $\gamma^{(0)}_{Sg,\,N}(\alpha_S) = 2T_R N_f \alpha_S/3\pi$ is the lowest-order term in the expansion (20).

The relation between the new anomalous dimensions $\gamma_{ab,\,N}^{(\mathrm{SDIS})}$ and those in the DIS scheme is the following (I drop the explicit dependence on N, α_S)

$$\gamma_{gg}^{(\mathrm{SDIS})} = \gamma_{gg} + \left[\frac{C_F}{C_A}(\gamma_{Sg} - \gamma_{Sg}^{(0)}) - \beta_0\,\alpha_S^2 \frac{\partial}{\partial\alpha_S} \ln\frac{\gamma_{Sg}}{\gamma_{Sg}^{(0)}}\right] + \mathcal{O}\Big(\alpha_S^2 \left(\frac{\alpha_S}{N}\right)^n\Big) \ , \tag{29}$$

$$\gamma_{gq}^{(\mathrm{SDIS})} = \gamma_{gq} + \left[\frac{\gamma_{Sg} - \gamma_{Sg}^{(0)}}{\gamma_{Sg}^{(0)}}\left(\gamma_{gq} - \frac{C_F}{C_A}\,\gamma_{gg}\right) - \frac{C_F}{C_A}\beta_0\,\alpha_S^2 \frac{\partial}{\partial\alpha_S} \ln\frac{\gamma_{Sg}}{\gamma_{Sg}^{(0)}}\right] + \mathcal{O}\Big(\alpha_S^2 \left(\frac{\alpha_S}{N}\right)^n\Big) \ , \tag{30}$$

$$\gamma_{Sq}^{(\mathrm{SDIS})} = \gamma_{Sq}^{(0)} + \mathcal{O}\left(\alpha_S^2\,(\alpha_S/N)^n\right) \ , \tag{31}$$

$$\gamma_{SS}^{(\mathrm{SDIS})} = \gamma_{SS}^{(0)} + \mathcal{O}\left(\alpha_S^2\,(\alpha_S/N)^n\right) \ , \tag{32}$$

where $12\pi\beta_0 = 11C_A - 2N_f$ is the first coefficient of the QCD β-function.

Note that, due to the colour charge relation (13), the terms in the square brackets on the right-hand side of Eqs. (29),(30) are of the order of $\alpha_S(\alpha_S/N)^n$, that is, these are next-to-leading contributions in resummed perturbation theory. Thus, to leading logarithmic accuracy, the evolution in the gluon sector is still controlled by the BFKL anomalous dimension in Eq. (10).

The main property of the SDIS scheme is represented by Eqs. (31),(32). We see that all the resummed contributions $\alpha_S(\alpha_S/N)^n$ discussed in the previous Section have been removed from the quark anomalous dimensions and absorbed, via Eq. (28), into the redefinition of the gluon density in Eq. (26).

The effects related to the resummation in Eq. (14) are now included in the gluon channel: partly in corrections of the order of $\alpha_S(\alpha_S/N)^n$ in the new gluon anomalous dimensions and partly in the x-dependent (N-dependent) normalization of the new gluon density. Note

---

[§]The scheme of DIS type discussed in Ref. [29] has $U_{gg,\,N}(\alpha_S) = R(\gamma_N(\alpha_S))$, where $R(\gamma)$ is the factor in Eq. (16).

that the factors $U_{gg,\,N}(\alpha_S)$ and $U_{gS,\,N}(\alpha_S)$ in Eqs. (27),(28) are power series of leading-order contributions $(\alpha_S/N)^n$¶. Moreover, these factors are proportional to the DIS scheme anomalous dimension $\gamma_{Sg,\,N}(\alpha_S)$. From the discussion in the previous Section, it follows that the gluon density $\tilde{f}_g^{(\mathrm{SDIS})}$ is much steeper than $\tilde{f}_g$ at small x.

In the SDIS scheme Eq. (1) remains unchanged ($\tilde{f}_S^{(\mathrm{SDIS})} = \tilde{f}_S$), whilst the master equation (2) becomes

$$\begin{aligned} \frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} &= \; < e_f^2 > \int_x^1 dz \left[P_{SS}^{(\mathrm{SDIS})}(\alpha_S(Q^2), z)\; \tilde{f}_S\left(x/z, Q^2\right) \right. \\ &+ \left. P_{Sg}^{(\mathrm{SDIS})}(\alpha_S(Q^2), z)\; \tilde{f}_g^{(\mathrm{SDIS})}\left(x/z, Q^2\right)\right] + \;\dots\; + O(1/Q^2)\;\; . \end{aligned} \tag{33}$$

The main feature of Eq. (33), following from Eqs. (31),(32), is that the splitting functions $P_{Sg}^{(\mathrm{SDIS})}$ and $P_{SS}^{(\mathrm{SDIS})}$ differ from their fixed-order perturbative expansions only by mild (at least, in principle) *next-to-next-to-leading* logarithmic corrections of the type $\frac{\alpha_S^3}{x}(\alpha_S \ln x)^n$. Therefore, in this scheme one can more safely carry out the analysis of the scaling violations of F_2 without performing any resummation of logarithmically enhanced terms in the quark splitting functions.

Having in mind this feature, it is interesting to come back to the MRS(G) analysis. Up to two-loop order, the splitting functions $P_{Sg}^{(\mathrm{SDIS})}, P_{SS}^{(\mathrm{SDIS})}$ differ slightly‖ from the DIS scheme functions P_{Sg}, P_{SS}. Therefore, from the viewpoint of resummed perturbation theory, the input densities extracted from the MRS(G) analysis can be interpreted as the parton densities in the SDIS scheme. It is suggestive that the MRS(G) gluon density is much steeper than the corresponding quark density (see, Eq. (8)), in (qualitative) agreement with the steepness induced by the factorization scheme transformation in Eqs. (25-28).

6. Summary and outlook

In this contribution I have presented some comments on the theoretical interpretation of the HERA data on the proton structure function $F_2(x,Q^2)$. In particular, I have tried to discuss whether the observed rise of F_2 at small x can be regarded as a signal of non-conventional QCD dynamics.

As a starting point, I recalled that the HERA data on F_2 can be succesfully described by conventional perturbative QCD in terms of (calculable) fixed-order Altarelli-Parisi splitting functions and quite steep (phenomenological) input parton densities [4-10].

Then, I reviewed how a non-conventional QCD approach at small x can be set up in terms of resummed Altarelli-Parisi splitting functions [16-22]. I also emphasized that within this approach one needs and does have [20,22] full control of the factorization scheme dependence of the parton densities. Hence, I recalled the explicit resummed results known at

¶The transformation matrix $U_{ab,\,N}(\alpha_S)$ that relates the DIS and the $\overline{\mathrm{MS}}$ schemes is instead of the order of $\alpha_S(\alpha_S/N)^n$ [22].

‖Actually, it is very trivial to modify the factor $U_{gg,\,N}(\alpha_S)$ in Eq. (28) in order to define a scheme, say SDIS' scheme, in which $P_{Sa}^{(\mathrm{SDIS}')}$ exactly coincides with P_{Sa} in the DIS scheme up to two-loop order. This scheme transformation is obtained by setting $U_{gg} = \gamma_{Sg}/(\gamma_{Sg}^{(0)} + \gamma_{Sg}^{(1)})$, where $\gamma_{Sg}, \gamma_{Sg}^{(0)}$ and $\gamma_{Sg}^{(1)}$ respectively are the resummed, one-loop and two-loop quark anomalous dimensions in the DIS scheme.

present, namely, the leading-order gluon splitting functions (or BFKL anomalous dimension) [16,20,28] and the next-to-leading-order quark splitting functions [21,22].

In spite of being the leading term in the resummation approach, the BFKL anomalous dimension by itself has a relatively weak impact on the phenomenology of the proton structure function in the HERA region. As a matter of fact, because of strong cancellations of $\ln x$-terms due to colour coherence [30], the BFKL anomalous dimension has a slow departure from its fixed-order perturbative expansion and the approach to its steep asymptotic behaviour is very much delayed.

The quark anomalous dimensions, instead, turn out to be particularly important for the analysis of the scaling violations of F_2. Indeed, since the exchanged off-shell vector boson couples directly to quarks and not to gluons, next-to-leading-order effects in the quark channel can overcome leading-order effects in the gluon channel. More precisely, the master equation (2), that controls the scaling violations, involves the products (convolutions) $P_{Sg} \otimes \tilde{f}_g$, $P_{SS} \otimes \tilde{f}_S$ of the *quark* splitting functions and the parton densities. Thus, steep parton densities in the conventional QCD approach can actually mimick the effect of small-x resummation in the quark channel.

In the common factorization schemes ($\overline{\mathrm{MS}}$, DIS), the resummed quark splitting functions $P_{Sg}(\alpha_S, x)$, $P_{SS}(\alpha_S, x)$ are much steeper** than the corresponding splitting functions evaluated to the first few orders in perturbation theory. Therefore they lead to stronger scaling violations. As a result, also the combined use of next-to-leading-order resummation and almost flat input densities [2] may accomodate the HERA data on F_2 with QCD. The numerical analyses presented in Refs. [13] and [14] point towards this direction.

Alternatively and equivalently, one can consider a different factorization scheme (the SDIS scheme introduced in Sect. 5) in which the resummation effects in the quark splitting functions are absorbed into the redefinition of the gluon density. The latter turns out to be steeper than the corresponding quark density. In such a scheme, the analysis of the scaling violations of F_2 is very similar to that in the conventional approach (i.e. one can neglect the resummation in the quark splitting functions). Therefore this picture offers a qualitative explanation of the results found by the MRS(G) analysis [8]: the MRS(G) partons with $\lambda_g > \lambda_S$ may be interpreted as the partons in the resummed SDIS scheme.

This discussion on the scheme dependence of the small-x behaviour of the parton densities may eventually appear as useless gymnastics with no physical content. After all, the parton densities are not physical observables and the final results for F_2 are unchanged. The point is that, from the HERA data on F_2, one would like to determine a universal set of parton densities to be used for predicting the high-energy behaviour of other cross sections. To this purpouse the parton densities have to be convoluted with partonic cross sections evaluated in the corresponding factorization scheme. Care has to be taken in the scheme dependence of these partonic cross sections: their small-x behaviour in resummed perturbation theory can be very much scheme dependent (the difference between the DIS scheme and the SDIS scheme quark splitting functions is an example of that).

**As discussed in Sect. 4, their steepness is due both to the BFKL dynamics (the factor R in Eq. (14)) and to the transverse momentum dynamics (the factor h_2 in Eq. (14)) of the subprocess $\gamma^* g^* \to q\bar{q}$ in Fig. 1.

In summary, HERA may have seen a (weak) signal of non-conventional small-x dynamics not in the steep rise of F_2 but *rather* in stronger scaling violations at moderate values of Q^2. More definite conclusions demand further phenomenological investigations and more accurate data on F_2 in a range of x and Q^2 as largest as possible.

Moreover, it could be helpful to have at our disposal data on other observables. In fact, one of the main difficulty in the F_2 analysis is that we have only two experimental inputs, namely $F_2(x, Q^2)$ and $\partial F_2(x, Q^2)/\partial \ln Q^2$. Using them and Eqs. (1),(2), we can determine the two relevant phenomenological outputs (quark and gluon densities) only provided that the theoretical framework is fixed. As soon as one introduces an extra degree of freedom, the theory (namely, resummed or not resummed splitting functions), the system becomes underconstrained. Sufficiently accurate data on observables, like the longitudinal structure function and heavy-flavour cross sections, for which we have fixed-order as well as resummed calculations [22,17-19], can overconstrain the present situation.

With the foreseen increasing precision of experimental data at small-x, one should not only supply improved predictions but also estimate their theoretical accuracy. High-energy factorization [17,22] provides a framework for combining consistently and unambiguously collinear factorization with small-x resummation. Therefore it is particularly suitable for estimating and comparing the relative reliability of the theoretical predictions based on conventional or non-conventional QCD dynamics. The practical feasibility of this program has been shown in Ref. [13], where all the next-to-leading $\ln x$-corrections known at present have been consistently matched with the complete (non-logarithmic) two-loop contributions. Further efforts (detailed comparison of different factorization schemes, more studies on the depencence on the input densities, estimate of subdominant effects) along these lines as well as the calculation of the next-to-leading $\ln x$-terms in the gluon anomalous dimensions are certainly warranted.

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After the completion of this paper, some phenomenological results in the SDIS scheme have been presented in Ref. [31].

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Figure captions

Figure 1: $k_\perp$-factorization diagram for F_2 : the (upper) off-shell quark box is coupled to the (lower) BFKL gluon distribution.